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Extended Filters of *MS*-Algebras

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Abstract

For a filter T of an *MS*-algebra $\mathfrak{L}$ and a subset Z of $\mathfrak{L}$, a new extension filter of T is introduced, denoted by $E_T(Z)$. Many properties of $E_T(Z)$ are investigated and the lattice structure of the set of all $E_T(Z)$ is studied. A new definition related to $E_T(Z)$ is presented, called fixed filters relative to a subset of $\mathfrak{L}$. A generalisation of $E_T(Z)$ is illustrated by introducing the concept of strong filters, notated by $\overline{E_T(Z)}$. The strong extension $\overline{E_T(Z)}$ is characterized by the intersection of all strong filters fixed relative to an ideal $\mathfrak{L} - \mathfrak{P}$ for a prime filter $\mathfrak{P}$ of $\mathfrak{L}$.

Keywords: bounded distributive lattice; *MS*-algebra; filter; ideal.

1 Introduction

The class **MS** of all *MS*-algebras was first considered by Blyth and Varlet [7]. Their basic goal was to define a common frame to study several similarities between de Morgan algebras and Stone algebras. The class **MS** is a subclass of Berman class $\mathbf{K_{1,1}}$ which introduced by Berman in [3]. The theory of filters such as d_L- filters [1] and β-filters [12] of *MS*-algebras has been studied by many authors. In the twenty-first century, many related structures to lattice theory and *MS*-algebras [10, 11] had a great concern.

In this article, the concept of $E_T(Z)$ of a filter T is introduced for a nonempty subset Z in an *MS*-algebra. We prove that $E_T(Z)$ is an extension filter of the filter T. A complete distributive lattice is formed by $E_T(Z)$. The concept of a fixed filter relative to a subset of an *MS*-algebra is illustrated. Equivalent conditions were set for the fixed filter relative to a subset of an *MS*-algebra. The definition of strong fixed filter relative to a subset is introduced. Furthermore, we characterize fixed filters in the terms of strong fixed filters relative to the set $\mathfrak{L} - \mathfrak{P}$ for a prime filter $\mathfrak{P}$.

2 Preliminaries

In the following, we give the basic background to make the paper consistent.

A bounded distributive lattice $(\mathfrak{L}; \vee, \wedge, ^\circ, 0, 1)$ together with a unary operation $\lambda \mapsto \lambda^\circ$ satisfying,

$$1^\circ = 0, \lambda \leq \lambda^{\circ\circ} \quad \text{and} \quad (\lambda \wedge \mu)^\circ = \lambda^\circ \vee \mu^\circ,$$

is called an *MS*-algebra. Each element in an *MS*-algebra satisfies the given equalities.

Proposition 2.1. [6] *Let $\mathfrak{L} \in$ **MS** and $\lambda, \mu \in \mathfrak{L}$. Then;*

(1) $(\lambda \vee \mu)^\circ = \lambda^\circ \wedge \mu^\circ$.

(2) $(\lambda \vee \mu)^{\circ\circ} = \lambda^{\circ\circ} \vee \mu^{\circ\circ}$.

(3) $(\lambda \wedge \mu)^{\circ\circ} = \lambda^{\circ\circ} \wedge \mu^{\circ\circ}$.

(4) $\lambda^{\circ\circ\circ} = \lambda^\circ$.

(5) $0^\circ = 1$.

For $T \subseteq \mathfrak{L}$, T is characterised as a filter provided that T is a sublattice of $\mathfrak{L}$ and if $\alpha \in T, \omega \in \mathfrak{L}$, then $\alpha \vee \omega \in T$. A prime filter $\mathfrak{P}$ is a proper filter satisfying that if $\omega, \tau \in \mathfrak{L}$ such that $\omega \vee \tau \in \mathfrak{P}$ then $\omega \in \mathfrak{P}$ or $\tau \in \mathfrak{P}$. Let $\omega \in \mathfrak{L}$. We set the notation $[\omega)$ for the principal filter of $\mathfrak{L}$ generated by ω and it is equivalent to the following $[\omega) = \{\alpha \in \mathfrak{L} : \alpha \geq \omega\}$. For a non empty subset $Z \subseteq \mathfrak{L}$, the filter $[Z)$ of $\mathfrak{L}$ generated by the set Z is defined by

$$[Z) = \Big\{\lambda \in \mathfrak{L} : \lambda \geq z_1 \wedge z_2 \wedge ... \wedge z_n \quad \text{for} \quad z_1, z_2, ..., z_n \in Z\Big\}.$$

Associating the lattice $\mathfrak{L}$ with the distributive property, the symbol $\mathfrak{F}(\mathfrak{L})$ stands for the lattice of all filters ordered by inclusion. Obviously, the filter $[1) = \{1\}$ is the smallest member of $\mathfrak{F}(\mathfrak{L})$. Also, $[0) = \mathfrak{F}$ is the largest member of $\mathfrak{F}(\mathfrak{L})$. We notate the class of all ideals by $\mathbb{I}(\mathfrak{L})$.

Theorem 2.1. *If $\mathfrak{L} \in \mathbf{MS}$ and $T, R \in \mathfrak{F}(\mathfrak{L})$. Then*

$$T \nabla R = \Big\{\lambda \vee \mu : \lambda \in T \quad and \quad \mu \in R\Big\},$$

is a member of $\mathfrak{F}(\mathfrak{L})$.

Proof. Obviously, $1 \in T\nabla R$. For $\lambda \in T$ and $\mu \in R$, suppose $\delta \in \mathfrak{L}$ such that $\delta \geq \lambda \vee \mu$. Therefore $\delta \in T$ and $\delta \in R$. Thus, $\delta = \delta \vee \delta \in T\nabla R$.

Let $\delta, \gamma \in T\nabla R$. Then there exist $\lambda_1, \lambda_2 \in T$ and $\mu_1, \mu_2 \in R$ such that $\delta = \lambda_1 \vee \mu_1$ and $\gamma = \lambda_2 \vee \mu_2$. We have that

$$\begin{aligned}
\delta \wedge \gamma &= (\lambda_1 \vee \mu_1) \wedge (\lambda_2 \vee \mu_2) \\
&= \Big[(\lambda_1 \vee \mu_1) \wedge \lambda_2\Big] \vee \Big[(\lambda_1 \vee \mu_1) \wedge \mu_2\Big] \\
&= \Big[(\lambda_1 \wedge \lambda_2) \vee (\mu_1 \wedge \lambda_2)\Big] \vee \Big[(\lambda_1 \wedge \mu_2) \vee (\mu_1 \wedge \mu_2)\Big] \in T\nabla R,
\end{aligned}$$

since $\lambda_1 \wedge \lambda_2 \in T$ and $\mu_1 \wedge \mu_2 \in R$. Hence $T\nabla R \in \mathfrak{F}(\mathfrak{L})$. □

Theorem 2.2. *For $\mathfrak{L} \in \mathbf{MS}$. Let $J, K \in \mathbb{I}(\mathfrak{L})$. Define*

$$J \vee_I K = \Big\{z; z \leq \lambda \wedge \mu : \lambda \in J \quad and \quad \mu \in K\Big\}.$$

Then $J \vee_I K \in \mathbb{I}(\mathfrak{L})$.

Proof. Clearly, $0 \in J \vee_I K$. For $\lambda \in J$ and $\mu \in K$, suppose $\delta \in \mathfrak{L}$, such that $\delta \leq \lambda \wedge \mu$. Obviously, $\delta \in J \vee_I K$. Let $\delta, \gamma \in J \vee_I K$. It follows that $\delta \leq \lambda_1 \wedge \mu_1$ and $\delta \leq \lambda_2 \wedge \mu_2$ for $\lambda_1, \lambda_2 \in J$ and $\mu_1, \mu_2 \in K$. Then,

$$\begin{aligned}
\delta \vee \gamma &\leq (\lambda_1 \wedge \mu_1) \vee (\lambda_2 \wedge \mu_2) \\
&\leq \Big[(\lambda_1 \wedge \mu_1) \vee \lambda_2\Big] \wedge \Big[(\lambda_1 \wedge \mu_1) \vee \mu_2\Big] \\
&\leq \Big[(\lambda_1 \vee \lambda_2) \wedge (\mu_1 \vee \lambda_2)\Big] \wedge \Big[(\lambda_1 \vee \mu_2) \wedge (\mu_1 \vee \mu_2)\Big] \in J \vee_I K,
\end{aligned}$$

since $\lambda_1 \vee \lambda_2 \in J$ and $\mu_1 \vee \mu_2 \in K$. □

Theorem 2.3. [9] *For $\mathfrak{L} \in \mathbf{MS}$. The set $\mathfrak{L} - \mathfrak{P} \in \mathbb{I}(\mathfrak{L})$ providing that $\mathfrak{P}$ is a prime filter of $\mathfrak{L}$.*

For details of *MS*-algebras, [2] highlighted many aspects of the variety **MS**. In [5], the subvarieties of **MS** were determined . Also, many constructions and substructures of *MS*-algebras were presented in [4, 8]. Throughout the paper we use the symbol $\mathfrak{L}$ for an *MS*-algebra.

3 Extended Filter

For $T \in \mathfrak{F}(\mathfrak{L})$ and a nonempty subset Z of $\mathfrak{L}$, define

$$E_T(Z) = \Big\{\lambda \in \mathfrak{L}\,; \quad \lambda \vee z^{\circ\circ} \in T \quad \text{for every} \quad z \in Z\Big\}.$$

Theorem 3.1. *For $T \in \mathfrak{F}(\mathfrak{L})$, the set $E_T(Z)$ is a filter of $\mathfrak{L}$ containing T.*

Proof. Obviously, $1 \in E_T(Z)$. Assume that $\lambda \in E_T(Z)$ and $\mu \in \mathfrak{L}$ satisfying $\lambda \leq \mu$. We have that $\mu \vee z^{\circ\circ} \geq \lambda \vee z^{\circ\circ}$. Therefore $\mu \vee z^{\circ\circ} \in T$. Then $\mu \in E_T(Z)$. Assume that $\lambda, \mu \in E_T(Z)$. Since $(\lambda \wedge \mu) \vee z^{\circ\circ} = (\lambda \vee z^{\circ\circ}) \wedge (\mu \vee z^{\circ\circ}) \in T$, then $\lambda \wedge \mu \in T$. Clearly, $T \subseteq E_T(Z)$. □

We call $E_T(Z)$ an extended filter of T. The following theorem encapsulates many characterisations of $E_T(Z)$.

Lemma 3.1. *Let $T \in \mathfrak{F}(\mathfrak{L})$. For any nonempty subset Z of $\mathfrak{L}$, we have;*

(1) *If Z is contained in the subset W, then $E_T(W) \subseteq E_T(Z)$.*

(2) *If R is a filter contains T, then $E_T(Z) \subseteq E_R(Z)$.*

(3) *If T contains each element of Z, then $E_T(Z) = \mathfrak{L}$.*

(4) *If $0 \in Z$, then $E_T(Z) = \mathfrak{L}$ implies that $Z \subseteq T$.*

(5) *If $T \subseteq Z$ and $z^{\circ\circ} = 0$ for some $z \in Z$, then $E_T(Z) \cap Z = T$.*

(6) *If $\alpha^{\circ\circ} = 0$ for some $\alpha \in E_T(Z)$, then $E_T(E_T(Z)) \cap E_T(Z) = T$.*

(7) *$E_T(Z) = E_T([Z))$.*

(8) *$E_{E_T(Z)}(W) = E_{E_T(W)}(Z)$.*

Proof.

(1) Assume that $Z \subseteq W$. If $\lambda \in E_T(W)$, then $\lambda \vee w^{\circ\circ} \in T$ for every $w \in W$. It follows that $\lambda \vee z^{\circ\circ} \in T$ for every $z \in Z \subseteq W$. Hence $\lambda \in E_T(Z)$.

(2) Suppose $T \subseteq R$ and $\lambda \in E_T(Z)$. This implies that $\lambda \vee z^{\circ\circ} \in T \subseteq R$ for every $z \in Z$. Thus $\lambda \in E_R(Z)$. Hence $E_T(Z) \subseteq E_R(Z)$.

(3) Let $Z \subseteq T$. Obviously, $E_T(Z) \subseteq \mathfrak{L}$. Conversely, suppose $\lambda \in \mathfrak{L}$ and $z \in Z$. Since $z \in T$ and $\lambda \vee z^{\circ\circ} \geq z^{\circ\circ} \geq z$, then $\lambda \vee z^{\circ\circ} \in T$. We conclude that $E_T(Z) = \mathfrak{L}$.

(4) Assume that $\lambda \in Z$ and $E_T(Z) = \mathfrak{L}$. Then $\lambda \vee z^{\circ\circ} \in T$ for every $z \in Z$. Hence $\lambda = \lambda \vee 0^{\circ\circ} \in T$.

(5) We have that $T \subseteq E_T(Z) \cap Z$. Conversely, let $\lambda \in E_T(Z) \cap Z$. We get that $\lambda \in Z$ and $\lambda \in E_T(Z)$. Then $\lambda \vee z^{\circ\circ} \in T$ for every $z \in Z$. Thus $\lambda = \lambda \vee 0 \in T$. This implies that $E_T(Z) \cap Z \subseteq T$. Hence $E_T(Z) \cap Z = T$.

(6) Follows directly from (5).

(7) By (1), $E_T([Z)) \subseteq E_T(Z)$. Conversely, suppose that $\lambda \in E_T(Z)$, then $\lambda \vee z^{\circ\circ} \in T$ for every $z \in Z$. Let $p \in [Z)$. It follows that $p \geq z_1 \wedge z_2 \wedge ... \wedge z_n$ for some $z_1, ..., z_n \in Z$. Then

$$\lambda \vee p^{\circ\circ} \geq \lambda \vee ({z_1}^{\circ\circ} \wedge ... \wedge {z_n}^{\circ\circ}) \geq (\lambda \vee {z_1}^{\circ\circ}) \wedge ... \wedge (\lambda \vee {z_n}^{\circ\circ}) \in T.$$

Hence $E_T(Z) = E_T([Z))$.

(8) We see that

$$\begin{aligned}\lambda \in E_{E_T(Z)}(W) &\iff \lambda \vee w^{\circ\circ} \in E_T(Z) \text{ for every } w \in W,\\ &\iff (\lambda \vee w^{\circ\circ}) \vee z^{\circ\circ} \in T \text{ for every } w \in W \text{ and } z \in Z,\\ &\iff (\lambda \vee z^{\circ\circ}) \vee w^{\circ\circ} \in T \text{ for every } z \in Z \text{ and } w \in W,\\ &\iff \lambda \vee z^{\circ\circ} \in E_T(W) \text{ for every } z \in Z,\\ &\iff \lambda \in E_{E_T(W)}(Z).\end{aligned}$$

□

Remark 3.1.

(1) *The converse of* (3) *is not necessarily true. For example, set* $\mathfrak{L} = \{0 \leq \lambda \leq \mu \leq \gamma \leq 1\}$, *such that* $\mu = \mu^\circ = \lambda^\circ, \gamma^\circ = 0 = 1^\circ,\ 0^\circ = 1$. *Clearly,* $(L, ^\circ) \in \mathbf{MS}$. *Suppose that* $T = [\mu) = \{\mu, \gamma, 1\}$ *and* $Z = \{\lambda, \gamma\} \nsubseteq T$. *Then* $E_T(Z) = \mathfrak{L}$. *So the condition* $0 \in Z$ *in* (4) *is necessary.*

(2) *The set* Z *is not necessarily a subset of* $E_T(E_T(Z))$. *For example, we obtain in the previous example that* $E_T(Z) = \mathfrak{L}$ *and* $E_T(E_T(Z)) = E_T(\mathfrak{L}) = T$.

For $T \in \mathfrak{F}(\mathfrak{L})$ and $Z \subseteq \mathfrak{L}$, we use the following notations :

$$\mathbb{E}(Z) = \Big\{E_T(Z);\quad T \in \mathfrak{F}(\mathfrak{L})\Big\},$$
$$\mathbb{E}_T = \Big\{E_T(Z);\quad Z \subseteq \mathfrak{L}\Big\}.$$

Proposition 3.1. *If* $T \in \mathfrak{F}(\mathfrak{L})$, *then* T *is a member of* $\mathbb{E}_T$. *Moreover,* T *is the smallest element in* $\mathbb{E}_T$.

Proof. It is easy to prove that $T = E_T(\{0\})$, thus $T \in \mathbb{E}_T$. Also, for every non empty $Z \subseteq \mathfrak{L}$ we have that $T \subseteq E_T(Z)$. Hence T is the smallest element in $\mathbb{E}_T$. □

In the next lemma, basic properties of $\mathbb{E}(Z)$ and $\mathbb{E}_\mathbb{T}$ are investigated seeking for constructing a new lattice.

Lemma 3.2. *Let* $\mathfrak{L} \in \mathbf{MS}$. *For nonempty subsets* Z *and* W *of* $\mathfrak{L}$, *we have;*

(1) $\bigcup_{\iota\in I} E_T(Z_\iota) \subseteq E_T(\bigcap_{\iota\in I} Z_\iota)$.

(2) $E_T(\bigcup_{\iota\in I} Z_\iota) \subseteq \bigcap_{\iota\in I} E_T(Z_\iota)$.

(3) $E_T(Z) \overline{\vee} E_T(W) = E_T(Z \cup W)$.

Proof.

(1) Obviously, $\bigcap_{\iota\in I} Z_\iota \subseteq Z_\iota$ for every $\iota \in I$. Then $E_T(Z_\iota) \subseteq E_T(\bigcap_{\iota\in I} Z_\iota)$ for every $\iota \in I$. Hence, $\bigcup_{\iota\in I} E_T(Z_\iota) \subseteq E_T(\bigcap_{\iota\in I} Z_\iota)$.

(2) Clearly, $Z_\iota \subseteq \bigcup_{\iota\in I} Z_\iota$ for every $\iota \in I$. By Lemma 3.1 (1), $E_T(\bigcup_{\iota\in I} Z_\iota) \subseteq E_T(Z_\iota)$ for every $\iota \in I$. Hence, $E_T(\bigcup_{\iota\in I} Z_\iota) \subseteq \bigcap_{\iota\in I} E_T(Z_\iota)$.

(3) Let $\lambda \in E_T(Z \cup W)$. Then $\lambda \vee \mu^{\circ\circ} \in T$ for every $\mu \in Z \cup W$. Thus, $\lambda \vee z^{\circ\circ} \in T$ and $\lambda \vee w^{\circ\circ} \in T$ and for every $z \in Z$ and every $w \in W$. So, $\lambda \in E_T(Z)$ and $\lambda \in E_T(W)$. This implies that $\lambda \in E_T(Z) \triangledown E_T(W)$.

Conversely, assume that $e \in E_T(Z) \triangledown E_T(W)$. Then, $e = \lambda \vee \mu$ for some $\lambda \in E_T(Z)$ and $\mu \in E_T(W)$. Let $z \in Z$ and $w \in W$. Then,

$$\begin{aligned} e \vee z^{\circ\circ} &= \big(\lambda \vee \mu\big) \vee z^{\circ\circ} \\ &= \mu \vee \big(\lambda \vee z^{\circ\circ}\big) \in T \quad \text{since} \quad \lambda \vee z^{\circ\circ} \in T. \end{aligned}$$

Similarly, $e \vee w^{\circ\circ} \in T$. Hence $\lambda \in E_T\big(Z \cup W\big)$.

□

Theorem 3.2. *Let $T \in \mathfrak{F}(\mathfrak{L})$. Let Z and W be nonempty subsets of $\mathfrak{L}$. Then;*

(1) $E_T(\mathfrak{L}) = T = E_T\big(\{0\}\big)$.

(2) $E_T\big(\{1\}\big) = \mathfrak{L} = E_T(T)$.

(3) $E_T(Z) \cap E_T(W) = E_T\big(Z \cup W\big)$.

(4) *If* $E_T\big(Z \cap W\big) \subseteq E_T(Z) \triangledown E_T(W)$, *then* $E_T\big(Z \cap W\big) = E_T(Z) \triangledown E_T(W)$.

Proof.

(1) We have that $T \subseteq E_T(\mathfrak{L})$. On the other hand, let $\lambda \in E_T(\mathfrak{L})$. Thus $\lambda \vee a^{\circ\circ} \in T$ for every $a \in \mathfrak{L}$. Since $\mathfrak{L}$ is bounded, we get that $\lambda = \lambda \vee 0^{\circ\circ} \in T$. We can easily see that $E_T(\{0\}) = T$.

(2) Clearly, $\mathfrak{L} = E_T(T)$. We only need to prove that $\mathfrak{L} = E_T(\{1\})$. Assume that $\lambda \in \mathfrak{L}$, then $\lambda \vee 1^{\circ\circ} = 1 \in T$.

(3) $Z, W \subseteq Z \cup W$ imply that $E_T(Z \cup W) \subseteq E_T(Z)$ and $E_T(Z \cup W) \subseteq E_T(W)$. So, $E_T(Z \cup W) \subseteq E_T(Z) \cap E_T(W)$. Conversely, let $\lambda \in E_T(Z) \cap E_T(W)$. Then $\lambda \in E_T(Z)$ and $\lambda \in E_T(W)$. It follows that $\lambda \vee z^{\circ\circ} \in T$ for every $z \in Z$ and $\lambda \vee w^{\circ\circ} \in T$ for every $w \in W$. Therefore $\lambda \vee a^{\circ\circ} \in T$ for every $a \in Z \cup W$. Hence $\lambda \in E_T(Z \cup W)$.

(4) Since $Z \cap W \subseteq Z, W$, then $E_T(Z), E_T(W) \subseteq E_T(Z \cap W)$. Hence, $E_T(Z) \triangledown E_T(W) \subseteq E_T(Z \cap W)$.

□

Corollary 3.1. *For $T \in \mathfrak{F}(\mathfrak{L})$. Assume that $E_T(Z \cap W) \subseteq E_T(Z) \triangledown E_T(W)$ for any two subsets Z and W of $\mathfrak{L}$. Then $\big(E_T;^{\triangledown},^{\wedge}, E_T(\{0\}), E_T(\{1\})\big)$ is a bounded distributive lattice.*

Remark 3.2. *Obviously, if $\mathfrak{L}$ is a complete lattice, then $\big(E_T;^{\triangledown},^{\wedge}, E_T(\{0\}), E_T(\{1\})\big)$ is also a complete lattice.*

Theorem 3.3. *If Z is a subset of $\mathfrak{L}$, then $\big(\mathbb{E}(Z);^{\triangledown},^{\wedge}, E_{[1)}(Z), E_{[0)}(Z)\big)$ is a bounded distributive lattice. Moreover, $\mathbb{E}(Z)$ is a complete lattice providing that $\mathfrak{L}$ is a complete lattice.*

Proof. For a subset Z of $\mathfrak{L}$ and $T \in \mathfrak{F}(\mathfrak{L})$, we show that $E_{\bigcap_{\iota\in I} T_\iota}(Z) = \bigcap_{\iota\in I} E_{T_\iota}(Z)$. We have that $\bigcap_{\iota\in I} T_\iota \subseteq T_\iota$ for every $\iota \in I$. By Lemma 3.1 (2), $E_{\bigcap_{\iota\in I} T_\iota}(Z) \subseteq E_{T_\iota}(Z)$ for every $\iota \in I$. Then $E_{\bigcap_{\iota\in I} T_\iota}(Z) \subseteq \bigcap_{\iota\in I} E_{T_\iota}(Z)$.

Conversely, let $\lambda \in \bigcap_{\iota\in I} E_{T_\iota}(Z)$. Then $\lambda \in E_{T_\iota}(Z)$ for every $\iota \in I$. This implies that $\lambda \vee z^{\circ\circ} \in T_\iota$ for every $z \in Z$ for every $\iota \in I$. Then $\lambda \vee z^{\circ\circ} \in \bigcap_{\iota\in I} T_\iota$ for every $z \in Z$. Therefore $\lambda \in E_{\bigcap_{\iota\in I} T_\iota}(Z)$. Hence $E_{\bigcap_{\iota\in I} T_\iota}(Z) = \bigcap_{\iota\in I} E_{T_\iota}(Z)$.

We also need to show that $E_{T\overline{\vee}R}(Z) = E_T(Z)\overline{\vee}E_R(Z)$. By Theorem 2.1 , we have that $E_T(Z)\overline{\vee}E_R(Z) = \{\lambda\vee\mu;\ \lambda \in E_T(Z),\ \mu \in E_T(Z)\}$ is a filter of $\mathfrak{L}$. For every $t \in T$ and $r \in R$ we have that $t, r \leq t \vee r$. Then $t \vee r \in T, R$. Therefore $T\overline{\vee}R \subseteq T, R$ and then $E_{T\overline{\vee}R}(Z) \subseteq E_T(Z), E_R(Z)$. Thus $E_{T\overline{\vee}R}(Z) \subseteq E_T(Z)\overline{\vee}E_R(Z)$. Conversely, assume that $e \in E_T(Z)\overline{\vee}E_R(Z)$, therefore $e = \lambda \vee \mu$ for some $\lambda \in E_T(Z)$ and $\mu \in E_R(Z)$, then for every $z \in Z$ we have,

$$\begin{aligned} e \vee z^{\circ\circ} &= \big(\lambda \vee \mu\big) \vee z^{\circ\circ} \\ &= \big(\lambda \vee z^{\circ\circ}\big) \vee \big(\mu \vee z^{\circ\circ}\big) \in T\overline{\vee}R. \end{aligned}$$

Hence $e \in E_{T\overline{\vee}R}(Z)$. If $\mathfrak{L}$ is a complete, then $\big(E_T;^{\overline{\vee}},^{\wedge}, E_T(\{0\}), E_T(\{1\})\big)$ is complete. □

Definition 3.1. *A filter T of $\mathfrak{L}$ is said to be fixed relative to a subset Z of $\mathfrak{L}$ if $E_T(Z) = T$.*

We denote the class of all fixed filters relative to subset Z of $\mathfrak{L}$ by Δ_Z. The following example illustrates the concept of a fixed filter relative to a subset of $\mathfrak{L}$.

Example 3.1. *Let $\mathfrak{L} = \{0 \leq \mu \leq \delta \leq 1\}$ such that $\mu = \mu^\circ$, $\delta^\circ = 0 = 1^\circ$, $0^\circ = 1$. Obviously, $(\mathfrak{L},^\circ) \in$ **MS**. Suppose that $T = [\mu) = \{\mu, \delta, 1\}$ and $Z = \{\mu, 0\}$. Obviously, $E_T(Z) = T$. Then T is fixed relative to Z. Suppose that $C = \{\delta\}$. Thus $E_T(C) = \mathfrak{L}$. Hence T is not fixed relative to C.*

Proposition 3.2. *Let $T \in \mathfrak{F}(\mathfrak{L})$ and $Z \in \mathfrak{L}$. The following statements are equivalent:*

(1) *If $\lambda^{\circ\circ} = 0$ for some $\lambda \in E_T(Z)$, then $E_T(E_T(Z)) = \mathfrak{L}$.*

(2) *T is fixed relative to a subset Z.*

(3) *T is fixed relative to a subset $[Z)$.*

Proof. By Lemma 3.1 (7), $E_T(Z) = T$ is equivalent to $E_T([Z)) = T$. Then (2) if and only if (3). Assume the condition of (2). We get that $E_T(E_T(Z)) = E_T(T) = \mathfrak{L}$. Thus (2) implies (1). Consider(1). Therefore $E_T(E_T(Z)) = \mathfrak{L}$. By Lemma 3.1 (6), $\mathfrak{L} \cap E_T(Z) = T$. Thus $E_T(Z) = T$. Hence, (1) implies (2). □

Proposition 3.3. *For a maximal filter M of $\mathfrak{L}$, M is fixed relative to Z provided that $E_M(Z)$ is a proper filter of $\mathfrak{L}$.*

Proof. Since $M \subseteq E_M(Z)$ and $E_M(Z) \neq \mathfrak{L}$, then $M = E_M(Z)$. □

Proposition 3.4. *Let $T \in \mathfrak{F}(\mathfrak{L})$ and let $Z, W \subseteq \mathfrak{L}$. If $Z \subseteq W$ and T is fixed relative to Z, then T is fixed relative to W.*

Proof. Let $Z \subseteq W$. Then $E_T(W) \subseteq E_T(Z) = T$. Therefore $E_T(W) = T$. Hence T is fixed relative to W. □

Proposition 3.5. *For $Z \subseteq \mathfrak{L}$, the set Δ_Z is a meet semi lattice of $\big(\mathbb{E}(Z), \bigcap\big)$.*

Proof. Clearly, Δ_Z is an ordered subset of $\mathbb{E}(Z)$ by restricting the relation $\leq$ to Δ_Z. By Theorem 3.3 , $E_{T\cap R}(Z) = E_T(Z) \cap E_R(Z) = T \cap R \in \Delta_Z$ for $T, R \in \Delta_Z$. □

4 Strong Extensions

In this section, we go further by defining the concept of strong fixed filter $\overline{E_T(\kappa)}$ relative to an ideal κ of an *MS*-algebra. We notate the class of all prime filters by **Spec**$(\mathfrak{L})$. For $T \in \mathfrak{F}(\mathfrak{L})$, define

$$\overline{E_T(\kappa)} = \Big\{\alpha \in \mathfrak{L} : \alpha \vee a^{\circ\circ} \in T, \quad \text{for some} \quad a \in \kappa\Big\},$$

for an ideal κ. Obviously, $E_T(\kappa) \subseteq \overline{E_T(\kappa)}$. So, we have $T \subseteq E_T(\kappa) \subseteq \overline{E_T(\kappa)}$. Thus $\overline{E_T(\kappa)}$ is an extension of both T and $E_T(\kappa)$.

Theorem 4.1. *If $\mathfrak{L} \in$ **MS**, $\kappa \in \mathbb{I}(\mathfrak{L})$ and $T \in \mathfrak{F}(\mathfrak{L})$. Then $\overline{E_T(\kappa)}$ is a filter of $\mathfrak{L}$.*

Proof. We see that $1 \in \overline{E_T(\kappa)}$, as $1 = 1 \vee 0^{\circ\circ}$. Assume that $\lambda \in \overline{E_T(\kappa)}$. Then $\lambda \vee a^{\circ\circ} \in T$ for some $a \in \kappa$. Let $\mu \in \mathfrak{L}$ satisfying that $\lambda \leq \mu$. Then $\mu \vee a^{\circ\circ} \geq \lambda \vee a^{\circ\circ} \in T$. Thus $\mu \in \overline{E_F(\kappa)}$. If $\lambda, \mu \in \overline{E_T(\kappa)}$, then $\lambda \vee a^{\circ\circ} \in T$ and $\mu \vee b^{\circ\circ} \in T$ for some $a, b \in \kappa$. We have

$$\begin{aligned}(\lambda \wedge \mu) \vee (a \vee b)^{\circ\circ} &= \big(\lambda \wedge \mu\big) \vee \big(a^{\circ\circ} \vee b^{\circ\circ}\big)\\ &= \Big[\big(\lambda \vee a^{\circ\circ}\big) \vee b^{\circ\circ}\Big] \wedge \Big[\big(\mu \vee b^{\circ\circ}\big) \vee a^{\circ\circ}\Big].\end{aligned}$$

Since $(\lambda \vee a^{\circ\circ}) \vee b^{\circ\circ} \geq \lambda \vee a^{\circ\circ}$ and $(\mu \vee b^{\circ\circ}) \vee a^{\circ\circ} \geq \mu \vee b^{\circ\circ}$. We conclude that $[(\lambda \vee a^{\circ\circ}) \vee b^{\circ\circ}] \wedge [(\mu \vee b^{\circ\circ}) \vee a^{\circ\circ}] \in T$. Hence $\lambda \wedge \mu \in \overline{E_T(\kappa)}$. □

The inclusion $E_T(\kappa) \subseteq \overline{E_T(\kappa)}$ is proper as shown in the next example.

Example 4.1. *Consider $\mathfrak{L}$ with the following Hasse diagram:*

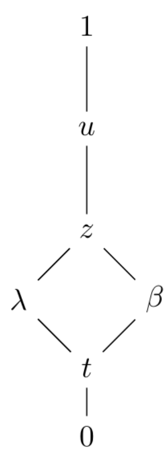

Figure 1: $\mathfrak{L}$.

Define a unary operation $^\circ$ *on* $\mathfrak{L}$ *by* $\lambda^\circ = t$, $\beta^\circ = z^\circ = t^\circ = u$, $u^\circ = \beta$, $1^\circ = 0$, $0^\circ = 1$. *Then* $(\mathfrak{L},^\circ) \in \mathbf{MS}$. *Take* $T = [\beta) = \{\beta, z, u, 1\}$ *and* $\kappa = (t] = \{0, t\}$. *Then,*

$$\begin{aligned} E_T((t]) &= \Big\{n \in \mathfrak{L} : n \vee 0^{\circ\circ} \in T \text{ and } n \vee t^{\circ\circ} \in T\Big\} \\ &= \Big\{n \in \mathfrak{L} : n \in T \text{ and } n \vee \beta \in T\Big\} \\ &= \Big\{\beta, z, u, 1\Big\} = [\beta). \end{aligned}$$

$$\begin{aligned} \overline{E_T((t])} &= \Big\{n \in \mathfrak{L} : n \vee 0^{\circ\circ} \in T \text{ or } n \vee t^{\circ\circ} \in T\Big\} \\ &= \Big\{n \in \mathfrak{L} : n \in T \text{ or } n \vee \beta \in T\Big\} \\ &= \Big\{0, t, \lambda, \beta, z, u, 1\Big\}. \end{aligned}$$

Lemma 4.1. *Let* $\mathfrak{L} \in \mathbf{MS}$, $T, R \in \mathfrak{F}(\mathfrak{L})$ *and* $\kappa, \kappa_1, \kappa_2 \in \mathbb{I}(\mathfrak{L})$. *Then;*

(1) $\kappa_1 \subseteq \kappa_2$ *implies that* $\overline{E_T(\kappa_1)} \subseteq \overline{E_T(\kappa_2)}$.

(2) $T \subseteq R$ *implies that* $\overline{E_T(\kappa)} \subseteq \overline{E_R(\kappa)}$.

(3) $\overline{E_T(\kappa)} \cap \overline{E_R(\kappa)} = \overline{E_{T\cap R}(\kappa)}$.

(4) $\overline{E_T(\kappa_1)} \cap \overline{E_T(\kappa_2)} = \overline{E_T(\kappa_1 \cap \kappa_2)}$.

(5) $\overline{E_T(\kappa)} = \overline{E_{\overline{E_T(\kappa)}}(\kappa)}$.

Proof.

(1) If $\alpha \in \overline{E_T(\kappa_1)}$, then $\alpha \vee a^{\circ\circ} \in T$ for some $a \in \kappa_1 \subseteq \kappa_2$. Thus $\alpha \in \overline{E_T(\kappa_2)}$.

(2) Suppose that $\lambda \in \overline{E_T(\kappa)}$. We get that $\lambda \vee a^{\circ\circ} \in T \subseteq R$ for some $a \in \kappa$. Consequently, $\lambda \in \overline{E_R(\kappa)}$.

(3) We have that $\overline{E_{T\cap R}(\kappa)} \subseteq \overline{E_T(\kappa)}$ and $\overline{E_{T\cap R}(\kappa)} \subseteq \overline{E_R(\kappa)}$. Since $T \cap R \subseteq T, R$, then $\overline{E_{T\cap R}(\kappa)} \subseteq \overline{E_T(\kappa)} \cap \overline{E_R(\kappa)}$. If $\lambda \in \overline{E_T(\kappa)} \cap \overline{E_R(\kappa)}$, then $\lambda \in \overline{E_T(\kappa)}$ and $\lambda \in \overline{E_R(\kappa)}$. Therefore $\lambda \vee a^{\circ\circ} \in T$ and $\lambda \vee b^{\circ\circ} \in R$ for some $a, b \in \kappa$. These imply that $\lambda \vee (a \vee b)^{\circ\circ} = \lambda \vee a^{\circ\circ} \vee b^{\circ\circ} \geq \lambda \vee b^{\circ\circ}, \lambda \vee a^{\circ\circ}$. Then $\lambda \vee (a \vee b)^{\circ\circ} \in T \cap R$. Thus $\lambda \in \overline{E_{T\cap R}(\kappa)}$. We conclude that $\overline{E_T(\kappa)} \cap \overline{E_R(\kappa)} = \overline{E_{T\cap R}(\kappa)}$.

(4) As $\overline{E_T(\kappa_1 \cap \kappa_2)} \subseteq \overline{E_T(\kappa_1)}, \overline{E_T(\kappa_2)}$, then $\overline{E_T(\kappa_1 \cap \kappa_2)} \subseteq \overline{E_T(\kappa_1)} \cap \overline{E_T(\kappa_2)}$. Conversely, let $\lambda \in \overline{E_T(\kappa_1)} \cap \overline{E_T(\kappa_2)}$. Then $\lambda \in \overline{E_T(\kappa_1)}$ and $\lambda \in \overline{E_T(\kappa_2)}$. It follows that $\lambda \vee a^{\circ\circ} \in T$ for some $a \in \kappa_1$ and $\lambda \vee b^{\circ\circ} \in T$ for some $b \in \kappa_2$. Then,

$$\lambda \vee (a \wedge b)^{\circ\circ} = (\lambda \vee a^{\circ\circ}) \wedge (\lambda \vee b^{\circ\circ}) \in T.$$

Since $a \wedge b \in \kappa_1 \cap \kappa_2$, then $\lambda \in \overline{E_T(\kappa_1 \cap \kappa_2)}$.

(5) Since $T \subseteq \overline{E_T(\kappa)}$, by (2), we get that $\overline{E_T(\kappa)} \subseteq \overline{E_{\overline{E_T(\kappa)}}(\kappa)}$. Conversely, let $\lambda \in \overline{E_{\overline{E_T(\kappa)}}(\kappa)}$. Then $\lambda \vee a^{\circ\circ} \in \overline{E_T(\kappa)}$ for some $a \in \kappa$. Therefore, $(\lambda \vee a^{\circ\circ}) \vee b^{\circ\circ} \in T$ for some $a, b \in \kappa$. Then $\lambda \vee (a \vee b)^{\circ\circ} \in T$. As $a \vee b \in \kappa$, we get that $\lambda \in \overline{E_T(\kappa)}$.

□

Lemma 4.2. *Let $T \in \mathfrak{F}(\mathfrak{L})$ and let Λ be a chain of members of $\mathbb{I}(\mathfrak{L})$. Then*

$$\overline{E_T(\bigcup_{\kappa\in\Lambda}\kappa)} = \bigcup_{\kappa\in\Lambda}\overline{E_T(\kappa)}.$$

Proof. Clearly, $\bigcup_{\kappa\in\Lambda}\kappa$ is an ideal of $\mathfrak{L}$. For each $\kappa \in \Lambda$, $\kappa \subseteq \bigcup_{\kappa\in\Lambda}\kappa$. This implies that $\overline{E_T(\kappa)} \subseteq \overline{E_T(\bigcup_{\kappa\in\Lambda}\kappa)}$. Then, $\bigcup_{\kappa\in\Lambda}\overline{E_T(\kappa)} \subseteq \overline{E_T(\bigcup_{\kappa\in\Lambda}\kappa)}$. Conversely, let $\lambda \in \overline{E_T(\bigcup_{\kappa\in\Lambda}\kappa)}$. Thus $\lambda \vee a^{\circ\circ} \in T$ for some $a \in \bigcup_{\kappa\in\Lambda}\kappa$. So, there exists $\kappa \in \Lambda$ such that $a \in \kappa$ and $\lambda \vee a^{\circ\circ} \in T$. Therefore $\lambda \in \overline{E_T(\kappa)}$ for some $\kappa \in \Lambda$. It follows that $\overline{E_T(\bigcup_{\kappa\in\Lambda}\kappa)} \subseteq \bigcup_{\kappa\in\Lambda}\overline{E_T(\kappa)}$. Hence the claim is true. □

Theorem 4.2. *If $T \in \mathfrak{F}(\mathfrak{L})$ and $\kappa \in \mathbb{I}(\mathfrak{L})$, then,*

$$\overline{E_T(\kappa)} = \bigcap\left\{\overline{E_T(\mathfrak{L}-\mathfrak{P})},\ \mathfrak{P} \in \textbf{\textit{Spec}}(\mathfrak{L}),\ \kappa \subseteq \mathfrak{L}-\mathfrak{P}\right\}.$$

Proof. We have $\overline{E_T(\kappa)} \subseteq \overline{E_T(\mathfrak{L}-\mathfrak{P})}$ for every $\mathfrak{P} \in \textbf{Spec}(\mathfrak{L})$.
Since $\kappa \subseteq \mathfrak{L}-\mathfrak{P}$, then $\overline{E_T(\kappa)} \subseteq \bigcap\{\overline{E_T(\mathfrak{L}-\mathfrak{P})} : \mathfrak{P} \in \textbf{Spec}(\mathfrak{L});\ \kappa \subseteq \mathfrak{L}-\mathfrak{P}\}$. On the other hand, by contrapositive we prove that $a \notin \overline{E_T(\kappa)}$ implies that $a \notin \bigcap\{\overline{E_T(\mathfrak{L}-\mathfrak{P})} : \mathfrak{P} \in \textbf{Spec}(\mathfrak{L}), \kappa \subseteq \mathfrak{L}-\mathfrak{P}\}$. Consider $\Gamma = \{J \in \mathbb{I}(\mathfrak{L}); \kappa \subseteq J$ and $a \notin \overline{E_T(J)}\}$. Obviously, $\kappa \in \Gamma$ so, $\Gamma \neq \phi$. Let Λ be a chain of members of Γ and $G = \bigcup_{J\in\Lambda} J$. By Lemma 4.2, $\overline{E_T(G)} = \bigcup_{J\in\Gamma}\overline{E_T(J)}$. Also, $\kappa \subseteq G$. Let $a \notin \overline{E_T(\kappa)}$.

We show that there exists $\mathfrak{P}_\circ \in \textbf{Spec}(\mathfrak{L})$ satisfying that $\kappa \subseteq \mathfrak{L}-\mathfrak{P}_\circ$ and $a \notin \overline{E_T(\mathfrak{L}-\mathfrak{P}_\circ)}$. Now, $a \notin \overline{E_T(J)}$ for all $J \in \Lambda$, implies that $a \notin \overline{E_T(G)}$. Therefore $\overline{E_T(G)}$ is an upper bound of Λ. By Zorn's Lemma, Γ has a maximal element $J_\circ$. Then $a \notin \overline{E_T(J_\circ)}$. So, $\mathfrak{L} \neq \overline{E_T(J_\circ)}$. Equivalently, $J_\circ \neq \mathfrak{L}$. Consider $\mathfrak{P}_\circ = \mathfrak{L} - J_\circ$. We show that $\mathfrak{P}_\circ \in \textbf{Spec}(\mathfrak{L})$. Clearly $1 \in \mathfrak{P}_\circ$. Let $\lambda \in \mathfrak{P}_\circ$ and $\mu \geq \lambda$. Then $\mu \notin J_\circ$. So, $\mu \in \mathfrak{P}_\circ$. Suppose that $\lambda, \mu \in \mathfrak{P}_\circ$. This implies that $\lambda^{\circ\circ}, \mu^{\circ\circ} \notin J_\circ$. So, $J_\circ \subseteq (J_\circ \cup \{\lambda^{\circ\circ}\}]$. Since $J_\circ$ is a maximal element of Γ, then $(J_\circ \cup \{\lambda^{\circ\circ}\}] \notin \Gamma$. We have $\kappa \subseteq J_\circ \subseteq (J_\circ \cup \{\lambda^{\circ\circ}\}] \nsubseteq \Gamma$ and $\kappa \subseteq J_\circ \subseteq (J_\circ \cup \{\mu^{\circ\circ}\}] \nsubseteq \Gamma$. Therefore,

$$a \in \overline{E_T(J_\circ \cup \{\lambda^{\circ\circ}\}]}\bigcap\overline{E_T(J_\circ \cup \{\mu^{\circ\circ}\}]} = \overline{E_T\Big((J_\circ \cup \{\lambda^{\circ\circ}\}]\bigcap(J_\circ \cup \{\mu^{\circ\circ}\}]\Big)}.$$

It follows that there exists $b \in (J_\circ \cup \{\lambda^{\circ\circ}\}]\bigcap(J_\circ \cup \{\mu^{\circ\circ}\}]$ such that $a \vee b^{\circ\circ} \in T$. That is, there exist $\lambda_1, \mu_1 \in J_\circ$ such that $b \leq \lambda_1 \vee \lambda^{\circ\circ}$ and $b \leq \mu_1 \vee \mu^{\circ\circ}$. Let $z = \lambda_1 \vee \mu_1$. Then $z \in J_\circ$ and $b \leq z \vee \lambda^{\circ\circ}$, $b \leq z \vee \mu^{\circ\circ}$. Therefore $a \vee b^{\circ\circ} \leq a \vee z^{\circ\circ} \vee \lambda^{\circ\circ}$ and $a \vee b^{\circ\circ} \leq a \vee z^{\circ\circ} \vee \mu^{\circ\circ}$. It follows that $a \vee z^{\circ\circ} \vee \lambda^{\circ\circ}, a \vee z^{\circ\circ} \vee \mu^{\circ\circ} \in T$. We get directly that $(a \vee z^{\circ\circ} \vee \lambda^{\circ\circ}) \wedge (a \vee z^{\circ\circ} \vee \mu^{\circ\circ}) \in T$. Then $(a \vee z^{\circ\circ}) \vee (\lambda^{\circ\circ} \wedge \mu^{\circ\circ}) \in T$. Thus $\lambda \wedge \mu \in \mathfrak{P}_\circ$. Otherwise, if $\lambda \wedge \mu \notin \mathfrak{P}_\circ$, then $\lambda \wedge \mu \in J_\circ$ implies that $a \vee z^{\circ\circ} \in \overline{E_T(J_\circ)} = E_{\overline{E_T(J_\circ)}}(J_\circ)$. Therefore, $a \in \overline{E_F(J_\circ)}$, which is a contradiction. Then $\mathfrak{P}_\circ$ is a filter.

It remains to prove that $\mathfrak{P}_\circ$ is prime. If $a \vee b \in \mathfrak{P}_\circ$. Then $a \vee b \notin J_\circ$. Thus $a \notin J_\circ$ or $b \notin J_\circ$. We conclude that $a \in \mathfrak{P}_\circ$ or $b \in \mathfrak{P}_\circ$. Thus $\mathfrak{P}_\circ$ is prime.

Moreover, $\kappa \subseteq \kappa_\circ = \mathfrak{L} - \mathfrak{P}_\circ$ and $a \notin \overline{E_T(J_\circ)} = \overline{E_{\mathfrak{F}}(\mathfrak{L}-\mathfrak{P}_\circ)}$. Hence, $\mathfrak{P}_\circ \in \textbf{Spec}(\mathfrak{L})$. Therefore, $\bigcap\{\overline{E_{\mathfrak{F}}(\mathfrak{L}-\mathfrak{P})}, \mathfrak{P} \in \textbf{Spec}(\mathfrak{L})\ , \kappa \subseteq \mathfrak{L}-\mathfrak{P}\} \subseteq \overline{E_T(\kappa)}$ and the proof is complete. □

Corollary 4.1. *Let $\mathfrak{L} \in$ **MS**, $\mu \in \mathfrak{L}$ and $T \in \mathfrak{F}(\mathfrak{L})$. Then,*

$$E_T(\{\mu\}) = \bigcap\left\{\overline{E_T(\mathfrak{L}-\mathfrak{P})},\quad \mathfrak{P} \in \textbf{\textit{Spec}}(\mathfrak{L}),\quad \mu \notin \mathfrak{P}\right\}.$$

Proof. We prove that $\overline{E_T((\mu])} = E_T(\{\mu\})$. Clearly, $E_T(\{\mu\}) \subseteq \overline{E_T((\mu])}$. Let $\lambda \in \overline{E_T((\mu])}$. Then there exists $b \in (\mu]$ such that $\lambda \vee b^{\circ\circ} \in T$. Thus $\lambda \vee b^{\circ\circ} \leq \lambda \vee \mu^{\circ\circ} \in T$. Then $\lambda \in E_T(\{\mu\})$. From Theorem 4.2, $\overline{E_T((\mu])} = \bigcap\{\overline{E_{\mathfrak{F}}(\mathfrak{L}-\mathfrak{P})}, \mathfrak{P} \in \textbf{Spec}(\mathfrak{L}), (\mu] \subseteq \mathfrak{L}-\mathfrak{P}\}$. It remains to prove that $(\mu] \subseteq \mathfrak{L}-\mathfrak{P}$ if and only if $\mu \notin \mathfrak{P}$.

$$\begin{aligned}(\mu] \subseteq \mathfrak{L}-\mathfrak{P} &\Longleftrightarrow \mu \in \mathfrak{L}-\mathfrak{P},\\ &\Longleftrightarrow \mu \notin \mathfrak{P}.\end{aligned}$$

Hence $E_T(\{\mu\}) = \bigcap\left\{\overline{E_T(\mathfrak{L}-\mathfrak{P})}, \quad \mathfrak{P} \in \textbf{Spec}(\mathfrak{L}), \quad \mu \notin \mathfrak{P}\right\}$. □

In Example 4.1, we show that the inclusion is proper. This motivates the following definition.

Definition 4.1. *A filter T of $\mathfrak{L}$ is said to be a strong fixed filter relative to an ideal I of $\mathfrak{L}$ if $T = \overline{E_T(\kappa)}$.*

Example 4.2. *Consider the MS-algebra in Example 3.1. Suppose that $T = [\delta) = \{\delta, 1\}$ and $\kappa = \{\mu, o\}$, thus $\overline{E_T(\kappa)} = T$. Hence T is a strong fixed filter relative to κ. Take $R = \{\mu, \delta, 1\}$. We have $\overline{E_R(\kappa)} = \mathfrak{L}$. Then R is not a strong fixed filter relative to κ.*

Proposition 4.1. *Every strong fixed filter relative to an ideal κ of $\mathfrak{L}$ is a fixed filter of $\mathfrak{L}$ relative to κ.*

Proof. Assume that T is a strong fixed filter relative to an ideal κ of $\mathfrak{L}$. Then $\overline{E_T(\kappa)} = T$. Since $T \subseteq E_T(\kappa) \subseteq \overline{E_T(\kappa)} = T$, then $T = E_T(\kappa)$. □

Proposition 4.2. *If $\kappa_1, \kappa_2 \in \mathbb{I}(\mathfrak{L})$ and T is a strong fixed filter of $\mathfrak{L}$ relative to κ_2 satisfying $\kappa_1 \subseteq \kappa_2$. Then T is a strong fixed filter of $\mathfrak{L}$ relative to κ_1.*

Proof. Suppose that T is a strong fixed filter of $\mathfrak{L}$ relative to an ideal κ_2. Then $T = \overline{E_T(\kappa_2)}$. We have $\kappa_1 \subseteq \kappa_2$. Therefore $\overline{E_T(\kappa_1)} \subseteq \overline{E_T(\kappa_2)} = T$. Also, $T \subseteq \overline{E_T(\kappa_1)}$. Hence $T = \overline{E_T(\kappa_1)}$. □

Remark 4.1. *A prime filter $\mathfrak{P}$ of $\mathfrak{L}$ is not necessarily a strong fixed filter of $\mathfrak{L}$ relative to the ideal $\mathfrak{L}-\mathfrak{P}$. Consider the following Hasse diagram $\mathfrak{L}$ in Figure 2. Define a unary operation $^\circ$ on $\mathfrak{L}$ by $\lambda^\circ = \eta^\circ = \eta$, $\delta^\circ = \nu^\circ = \delta$, $1^\circ = \gamma^\circ = \beta^\circ = \rho^\circ = 0$, $0^\circ = 1$. Then $(\mathfrak{L}, ^\circ) \in \mathbf{MS}$. Take $\mathfrak{P} = [\eta) = \{\eta, \rho, 1\}$. We have $\mathfrak{L}-\mathfrak{P} = \{0, \nu, \delta, \lambda, \beta, \gamma\}$. Then,*

$$\begin{aligned}\overline{E_{\mathfrak{P}}(\mathfrak{L}-\mathfrak{P})} &= \left\{u \in \mathfrak{L} : u \vee \gamma^{\circ\circ} \in \mathfrak{P} \quad \textit{for some} \quad \gamma \in \mathfrak{L}-\mathfrak{P}\right\}\\ &= \left\{u \in \mathfrak{L} : u \vee 0 \in \mathfrak{P}, \quad \textit{or} \quad u \vee \delta \in \mathfrak{P}, \quad \textit{or} \quad u \vee \eta \in \mathfrak{P}, \quad \textit{or} \quad u \vee 1 \in \mathfrak{P}\right\}\\ &= \mathfrak{L} \neq \mathfrak{P}.\end{aligned}$$

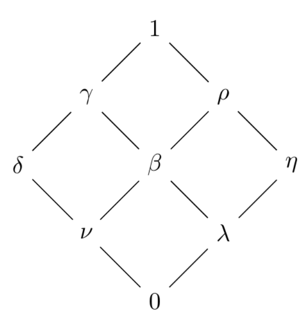

Figure 2: $\mathfrak{L}$.

Proposition 4.3. *If M is a maximal filter of $\mathfrak{L}$, then $\overline{E_M(\mathfrak{L}-M)} = \mathfrak{L}$ or M is a strong fixed filter of $\mathfrak{L}$ relative to the ideal $\mathfrak{L}-M$.*

Proof. We have proved that $\overline{E_M(\mathfrak{L}-M)}$ is a filter which contains M.
Then, either $\overline{E_M(\mathfrak{L}-M)} = M$ or $\overline{E_M(\mathfrak{L}-M)} = \mathfrak{L}$. □

Now, we study the lattice structure of the following sets

$$\overline{\mathbf{E}(\kappa)} = \left\{\overline{E_T(\kappa)}, \quad T \in \mathfrak{F}(\mathfrak{L})\right\},$$
$$\overline{\mathbf{E}_T} = \left\{\overline{E_T(\kappa)}, \quad \kappa \in \mathbb{I}(\mathfrak{L})\right\}.$$

Theorem 4.3. *Let κ be a proper ideal of an MS-algebra $\mathfrak{L}$. Then $\left(\overline{\mathbf{E}(\kappa)};^{\triangledown}, \bigcap, \overline{E_{\{1\}}(\kappa)}, \overline{E_{\mathfrak{L}}(\kappa)}\right)$ is a bounded distributive lattice by defining*

$$\overline{E_T(\kappa)}\triangledown\overline{E_R(\kappa)} = \overline{E_{T\triangledown R}(\kappa)},$$
$$\overline{E_T(\kappa)} \cap \overline{E_R(\kappa)} = \overline{E_{T\cap R}(\kappa)}.$$

Moreover, if $\mathfrak{L}$ is a complete lattice, then $\overline{\mathbf{E}(\kappa)}$ is complete.

Proof. The element $\overline{E_{\{1\}}(\kappa)}$ is the smallest in $\overline{\mathbf{E}(\kappa)}$. We have that $\{1\} \subseteq T$ for every $T \in \mathfrak{F}(\mathfrak{L})$. This implies that $\overline{E_{\{1\}}(\kappa)} \subseteq \overline{E_T(\kappa)}$. Also, $\overline{E_{\mathfrak{L}}(\kappa)}$ is the largest element in $\overline{\mathbf{E}(\kappa)}$, since $T \subseteq \mathfrak{L}$ for every $T \in \mathfrak{F}(\mathfrak{L})$, then $\overline{E_T(\kappa)} \subseteq \overline{E_{\mathfrak{L}}(\kappa)}$. In fact, $\overline{E_{\mathfrak{L}}(\kappa)} = \mathfrak{L}$ since for every $\alpha \in \mathfrak{L}$, $\alpha \vee 0^{\circ\circ} \in \mathfrak{L}$.

By Lemma 4.1, $\overline{E_T(\kappa)} \cap \overline{E_R(\kappa)} = \overline{E_{T\cap R}(\kappa)}$. It remains to prove that $\overline{E_T(\kappa)}\triangledown\overline{E_R(\kappa)} = \overline{E_{T\triangledown R}(\kappa)}$. We have that $f, g \leq f \vee g$ for every $f \in T$ and $g \in R$. Then $f \vee g \in T, R$. Thus $T\triangledown R \subseteq T, R$. This implies that $\overline{E_{T\triangledown R}(\kappa)} \subseteq \overline{E_T(\kappa)}, \overline{E_R(\kappa)}$. It follows that $\overline{E_{T\triangledown R}(\kappa)} \subseteq \overline{E_T(\kappa)}\triangledown\overline{E_R(\kappa)}$. On the other hand, let $z \in \overline{E_T(\kappa)}\triangledown\overline{E_R(\kappa)}$. Then $z = \lambda \vee \mu$ for some $\lambda \in \overline{E_T(\kappa)}$ and $\mu \in \overline{E_R(\kappa)}$. Then $\lambda \vee a^{\circ\circ} \in T$ for some $a \in \kappa$ and $\mu \vee b^{\circ\circ} \in R$ for some $b \in \kappa$. Hence,

$$\begin{aligned} z \vee (a \vee b)^{\circ\circ} &= \big(\lambda \vee \mu\big) \vee \big(a \vee b\big)^{\circ\circ} \\ &= \big(\lambda \vee a^{\circ\circ}\big) \vee \big(\mu \vee b^{\circ\circ}\big) \in T\triangledown R. \end{aligned}$$

We conclude that $z \in \overline{E_{T\triangledown R}(\kappa)}$. Then $\overline{E_T(\kappa)}\triangledown\overline{E_R(\kappa)} \subseteq \overline{E_{T\triangledown R}(\kappa)}$. The completeness of $\overline{\mathbf{E}(\kappa)}$ is immediate. □

Theorem 4.4. *If J and K are ideals of an MS-algebra $\mathfrak{L}$ and $\overline{E_T(J)} \vee_I \overline{E_T(K)} \subseteq \overline{E_T(J \vee_I K)}$, then $(\overline{\mathbf{E}_T};^{\vee_I}, \bigcap, \overline{E_T(\{0\})}, \overline{E_T(\mathfrak{L})})$ is a bounded distributive lattice.*

Proof. The element $\overline{E_T(\{0\})}$ is the smallest.
Since $\{0\} \subseteq J$ for every $J \in \mathbb{I}(\mathfrak{L})$, then $\overline{E_T(\{0\})} \subseteq \overline{E_T(J)}$. Also, $\overline{E_T(\mathfrak{L})}$ is the largest element in $\overline{\mathbf{E}_T}$.
Since $J \subseteq \mathfrak{L}$ for every $J \in \mathbb{I}(\mathfrak{L})$, then $\overline{E_T(J)} \subseteq \overline{E_T(\mathfrak{L})}$.

From Lemma 4.1, $\overline{E_T(J)} \cap \overline{E_T(K)} = \overline{E_T(J \cap K)}$. It remains to prove that $\overline{E_T(J)} \vee_I \overline{E_T(K)} = \overline{E_T(J \vee_I K)}$. Since $J \vee_I K \subseteq J, K$, then $\overline{E_T(J \vee_I K)} \subseteq \overline{E_T(J)}, \overline{E_T(K)}$, thus $\overline{E_T(J \vee_I K)} \subseteq \overline{E_T(J)} \vee_I \overline{E_T(K)}$.

By assumption, $\overline{E_T(J)}\vee_I\overline{E_T(K)} \subseteq \overline{E_T(J \vee_I K)}$. Hence, $(\overline{\mathbf{E}_T};^{\vee_I}, \bigcap, \overline{E_T(\{0\})}, \overline{E_T(\mathfrak{L})})$ is a bounded distributive lattice. □

5 Conclusions and Future work

In this paper, a new definition is presented and notated by $E_T(Z)$. We proved that $E_T(Z)$ is a filter containing T, consequently $E_T(Z)$ is called an extended filter of T. We concerned in studying a special type of extended filters called fixed filters. In fact, a fixed filter T is the smallest possible extended filter containing T with respect to a set.

A generalisation of $E_T(Z)$ was introduced by defining the strong extensions donated by $\overline{E_T(Z)}$. The extension $\overline{E_T(Z)}$ contains both T and $E_T(Z)$. We proved by a counter example that both $E_T(Z)$ and $\overline{E_T(Z)}$ are not the same. In future work, we may study the homomorphisms and topological spaces related to of $E_T(Z)$. Also, we can study the fuzzification of $E_T(Z)$.

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Conflicts of Interest The authors declare that they have no conflict of interest

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